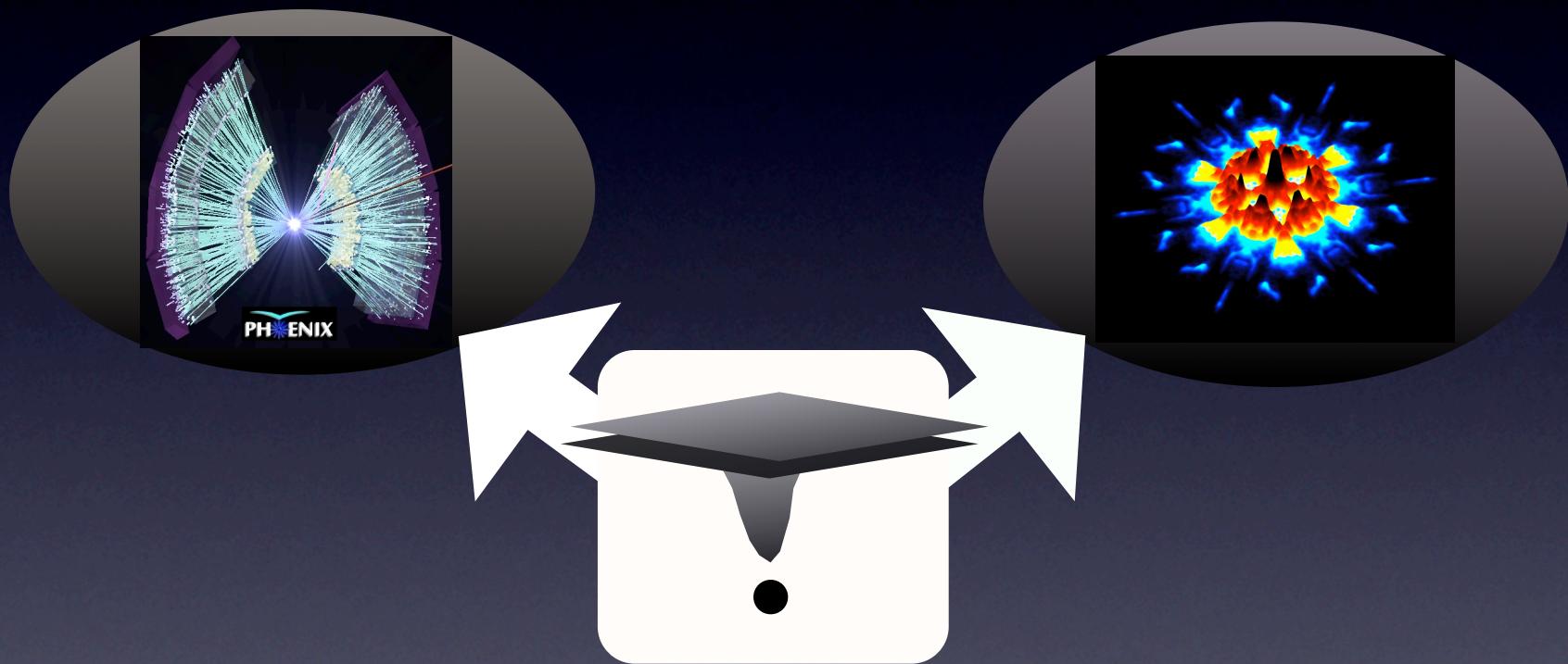




# Parity-Violating Hydrodynamics & Replacing the Entropy Argument

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Powerweek 2012, Universität Frankfurt, July 2012



by Matthias Kaminski (University of Washington)  
in collaboration with Jensen, Kovtun, Meyer, Ritz, Yarom

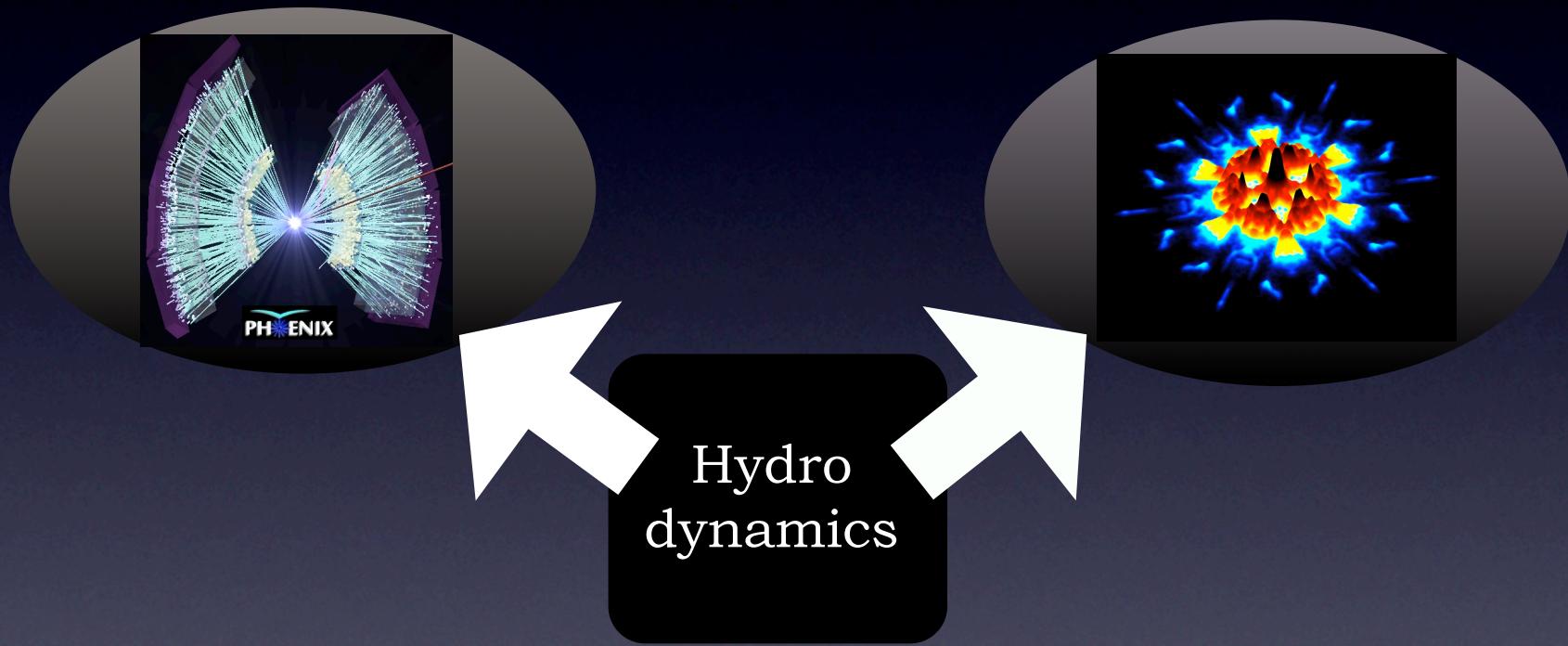
[arXiv:1112.4498]

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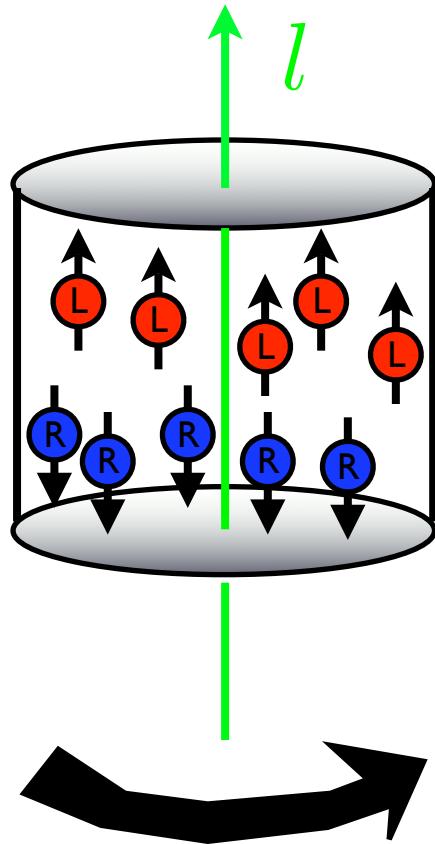
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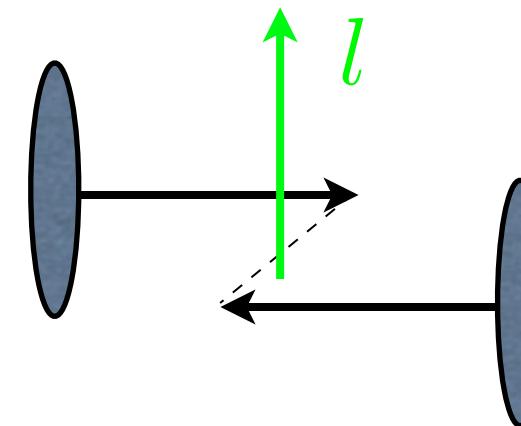
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# Invitation

*Chiral vortex effect*

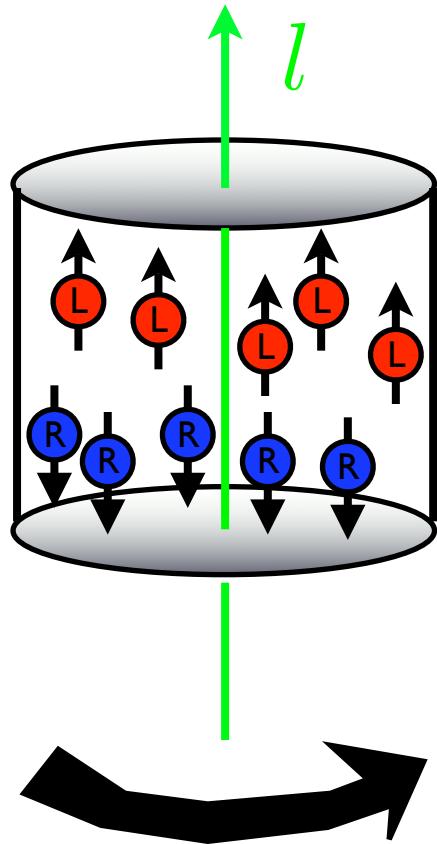


*Heavy-ion-collision*

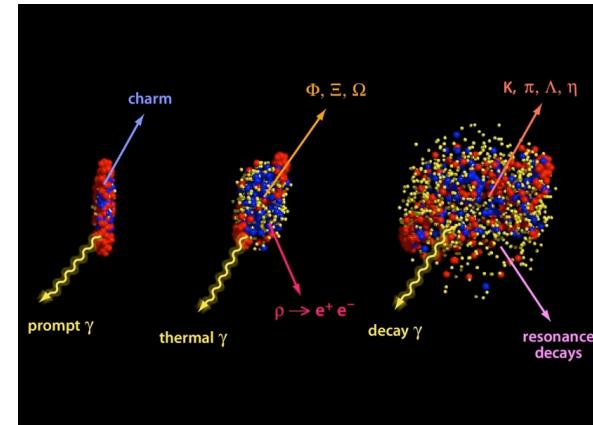
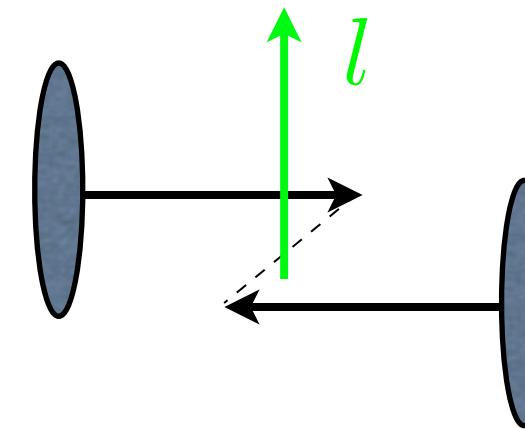


# Invitation

*Chiral vortex effect*



*Heavy-ion-collision*



# Invitation

*Fluid/Gravity Correspondence*

[Baier et al. 2007]

[Bhattacharyya et al. 0712.2456]

$$\text{Einstein equations} = \begin{array}{l} \text{hydrodyn. conservation} \\ \text{equations} \end{array} + \begin{array}{l} \text{dynamical} \\ \text{+ EOMs for} \\ \text{gravity fields} \end{array}$$



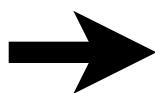
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Complete constitutive relation for EM-tensor, values for transport coefficients.  
(completes Israel-Stewart)



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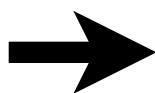
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## *Example: Fluid/Gravity derivation of chiral vortex effect.*

[Erdmenger, Haack, MK, Yarom 0809.2488]

[Banerjee et al. 0809.2596]

Computed all first/second order transport coefficients in a gravity dual without B.  
Gravity model: R-charged black hole with Chern-Simons term (induces anomaly).



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## *Pure field theory derivation.*

[Son, Surowka 0906.5044]



# Invitation

## *First order hydrodynamics*

Relativistic fluids with one conserved charge, with an anomaly (chiral)

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda \quad \nabla_\mu j^\mu = CE^\mu B_\mu$$

Constitutive equations

$$T^{\mu\nu} = \frac{\epsilon}{3}(4u^\mu u^\nu + g^{\mu\nu}) + \tau^{\mu\nu}$$

$$j^\mu = nu^\mu - \sigma T(g^{\mu\nu} + u^\mu u^\nu) \partial_\nu \left( \frac{\mu}{T} \right) + \xi \omega^\mu$$

$$\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$$

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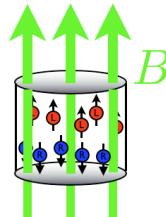
[Banerjee et al. 0809.2596]

New vorticity term arises!  
(related to triangle anomaly )

$$\xi = C \left( \mu^2 - \frac{2}{3} \frac{\mu^3 n}{\epsilon + P} \right)$$

[Son, Surowka 0906.5044]

See also chiral magnetic effect:



[Kharzeev et al., 2007]  
[Fukushima et al., 2008]



# Invitation

- New coefficient at first order hydrodynamics ( $\sim$ viscosity)
- $\xi$  completely determined by C and equation of state



# Invitation

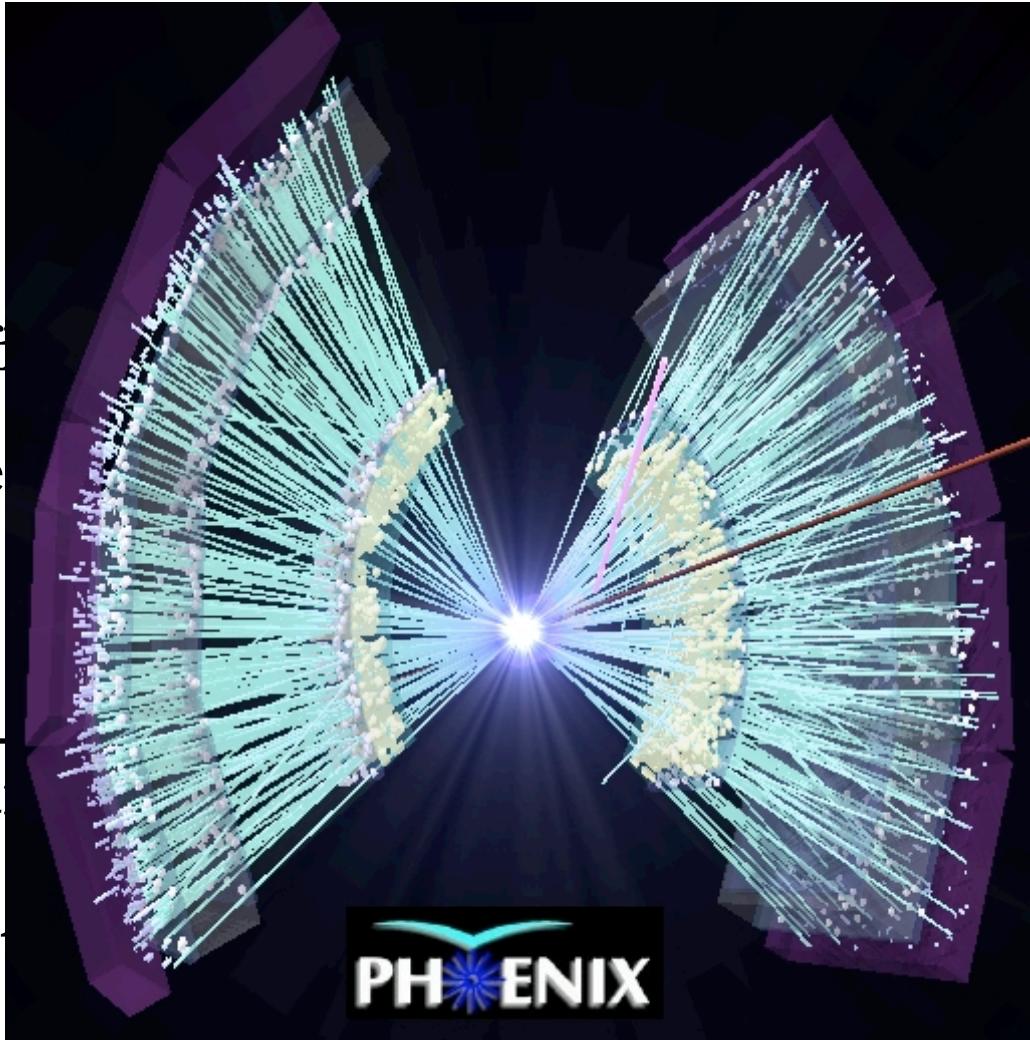
- New coefficient at first order hydrodynamics (~viscosity)
- $\xi$  completely determined by C and equation of state

- Relativistic hydrodynamics needs to be completed.  
*[Baier et al, Minwalla et al 2008]*
- Effects measured in heavy-ion-collisions?  
*[Kharzeev, Son]*
- Not in non-relativistic setups, so repeat for 2+1 dimensional QFT, with condensed matter applications in mind (parity anomaly?) *[Nicolis & Son, 1103.2137] [1112.4498]*



# Invitation

- New coefficients ( $\zeta$ )  
in terms of entropy (~viscosity)
- $\xi$  complex  
function of state



- Related work has been completed.  
[Minwalla et al 2008]
- Effect of boundary conditions?  
[Kharzeev, Son]
- Not in non-relativistic setups, so repeat for 2+1 dimensional QFT, with condensed matter applications in mind (parity anomaly?) [Nicolis & Son, 1103.2137] [1112.4498]



# **Invitation**

## *Preview of 2+1 dimensional results*

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \quad \nabla_\mu J^\mu = 0$$



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$$T^{\mu\nu} = \epsilon_0 u^\mu u^\nu + (P_0 - \zeta \nabla_\alpha u^\alpha - \tilde{\chi}_B B - \tilde{\chi}_\Omega \Omega) \Delta^{\mu\nu} - \eta \sigma^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu}$$

$$J^\mu = \rho_0 u^\mu + \sigma V^\mu + \tilde{\sigma} \tilde{V}^\mu + \tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T$$

“New” transport terms  
arise!



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“New” transport terms  
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thermodynamic  
interpretation of

$\tilde{\chi}_E$ ,  $\tilde{\chi}_\Omega$ ,  $\tilde{\chi}_B$

$\tilde{\eta}$  Hall viscosity

$\tilde{\chi}_E$ ,  $\tilde{\sigma}$  off-diagonal conductivity  
(anomalous Hall conductivity)

$\tilde{\chi}_T$  “thermal Hall conductivity”

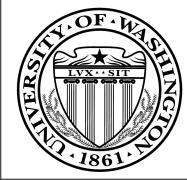
“New” method of restricting transport coefficients



# Outline

## ✓ Invitation

- I. Review: Hydrodynamics & parity breaking
- II. Parity-violating hydro in 2+1 dimensions
  - Classify transport coefficients
  - Restrictions from entropy production
  - Restrictions from two-point-functions
- III. One gravity dual
- IV. Conclusions



# I. Hydrodynamics & Parity-breaking

## *Basic idea*

- Introduce hydrodynamics language
- Sketch standard methods
- Summarize results in 3+1 dimensions



# I. Hydrodynamics & Parity-breaking

*First order conformal hydrodynamics in 3+1*

Hydrodynamics is an expansion in gradients (equivalently: low frequencies and large momenta).

Constitutive equations

$$T^{\mu\nu} = \frac{\epsilon}{3}(4u^\mu u^\nu + g^{\mu\nu}) + \tau^{\mu\nu}$$

$$j^\mu = nu^\mu - \sigma T \underbrace{(g^{\mu\nu} + u^\mu u^\nu)}_{\text{Vorticity}} \partial_\nu \left( \frac{\mu}{T} \right) + \xi \omega^\mu$$

$$\text{Vorticity } \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho =: \Delta^{\mu\nu}$$

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[Erdmenger, Haack, M.K., Yarom 0809.2488]

from writing down all possible terms (respecting symmetries) with one derivative, built from  $\{u, \epsilon, T, n, \mu, \epsilon^{\mu\nu\rho\dots}\}$ .

Examples     $\{\nabla^\nu \mu, \nabla^\nu T, nu^\nu, u^\nu u^\kappa \nabla_\kappa n, u^\nu n \nabla_\kappa u^\kappa, \dots\}$                           [Landau, Lifshitz]



# I. Hydrodynamics & Parity-breaking

## *First order traditional method*

1. Write down all first order (pseudo)vectors and (pseudo)tensors
2. Restricted by conservation equations

*Example: no external fields*

$$\nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu J^\mu = 0$$

$$0 = \nabla_\mu n u^\mu = n \nabla_\mu u^\mu + u^\mu \nabla_\mu n$$

Possibly restricted by conformal symmetry

*Example*  $\nabla^\nu \left( \frac{\mu}{T} \right)$  invariant under Weyl rescaling

3. Further restricted by positivity of entropy production

$$\nabla_\mu J_s^\mu \geq 0$$



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*Entropy argument  
[Landau, Lifshitz]*



# I. Hydrodynamics & Parity-breaking

*First order hydrodynamics in 3+1*

Restricted by entropy production argument

$$\nabla_\mu J_s^\mu \geq 0$$

*Example: Ideal current*       $J_s^\mu = s u^\mu$

$$D := u^\lambda \nabla_\lambda$$
$$\nabla \cdot u := \nabla_\lambda u^\lambda$$

$$\nabla_\mu J_s^\mu = s \nabla \cdot u + D s$$



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$$T D s = D \epsilon - \mu D n$$

$$0 = \nabla_\mu J^\mu = \nabla \cdot u + D n / n$$



$$= (s T \nabla \cdot u + D \epsilon + n \mu \nabla \cdot u) / T$$



# I. Hydrodynamics & Parity-breaking

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$$0 = u_\nu \nabla_\mu T^{\mu\nu} = -D\epsilon - (\epsilon + p) \nabla \cdot u \longrightarrow = [(sT + n\mu) \nabla \cdot u - (\epsilon + p) \nabla \cdot u] / T$$



# I. Hydrodynamics & Parity-breaking

## First order hydrodynamics in 3+1

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$$\nabla_\mu J_s^\mu \geq 0$$

Example: Ideal current  $J_s^\mu = su^\mu$

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$$\epsilon + p = sT + \mu n \longrightarrow = 0 \quad \text{no dissipation!}$$



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$$\epsilon + p = sT + \mu n \longrightarrow = 0 \quad \text{no dissipation!}$$

BUT: corrections generally carry entropy!

$$T^{\mu\nu} = T_{\text{ideal}}^{\mu\nu} + \tau^{\mu\nu}$$
$$J^\mu = J_{\text{ideal}}^\mu + \Upsilon^\mu$$



# I. Hydrodynamics & Parity-breaking

*First order non-conformal hydrodynamics in 3+1*

Most general entropy current

$$J_s^\lambda = su^\lambda - \frac{\mu}{T}\Upsilon^\lambda + s^\lambda$$

i) Parity-even sector

$$\nabla_\mu \left( su^\mu - \frac{\mu}{T} \Upsilon^\mu \right) = -\frac{1}{T} \nabla_\mu u_\nu \tau^{\mu\nu} - \Upsilon^\mu \left( \nabla_\mu \left( \frac{\mu}{T} \right) - \frac{E_\mu}{T} \right)$$
$$\Upsilon^\mu = -\sigma T \Delta^{\mu\nu} \partial_\nu \left( \frac{\mu}{T} \right) + \sigma E^\mu \quad s^\lambda = 0$$



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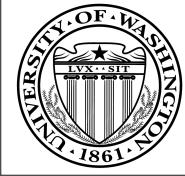
ii) Parity-odd sector

$$\nabla_\mu(su^\mu - \frac{\mu}{T}\Upsilon^\mu) = -\frac{1}{T}\nabla_\mu u_\nu \tau^{\mu\nu} - \Upsilon^\mu \left( \nabla_\mu(\frac{\mu}{T}) - \frac{E_\mu}{T} \right) - C \frac{\mu}{T} E \cdot B$$

$$\Upsilon^\mu = -\sigma T \Delta^{\mu\nu} \partial_\nu \left( \frac{\mu}{T} \right) + \sigma E^\mu + \xi \omega^\mu + \xi_B B^\mu$$

Most general additions:  $s^\mu = \alpha \omega^\mu + \alpha_B B^\mu$  [Son, Surowka 0906.5044]  
(also gravity dual)

$$\nabla_\mu J_s^\mu \geq 0 \implies \xi = C \left( \mu^2 - \frac{2}{3} \frac{n\mu^3}{\epsilon + P} \right), \quad \xi_B = C \left( \mu - \frac{1}{2} \frac{n\mu^2}{\epsilon + P} \right)$$



# I. Hydrodynamics & Parity-breaking

SUMMARY: (*non-conformal*) hydrodynamics in 3+1

[Son, Surowka 0906.5044]

Complete constitutive equations in 3+1 (with external gauge field)

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - (\zeta - \frac{2}{3}\eta)\Delta^{\mu\nu}\nabla_\gamma u^\gamma$$

$$j^\mu = nu^\mu + \sigma V^\mu + \xi\omega^\mu + \xi_B B^\mu$$

$$V^\mu = E^\mu - T\Delta^{\mu\nu}\nabla_\nu\left(\frac{\mu}{T}\right)$$

$$E^\mu = F^{\mu\nu}u_\nu$$

$$B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$$

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\nabla_\rho u_\sigma$$



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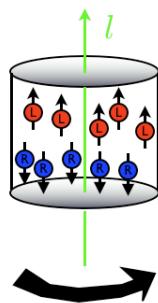
$$B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$$

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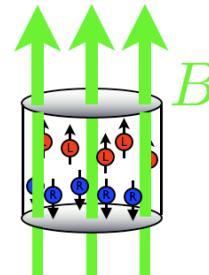
New transport coefficients restricted

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right)$$

Chiral  
vortex  
effect



Chiral  
magnetic  
effect



Observable in  
heavy-ion  
collisions

Predicted values:  
[Kharzeev, Son 1010.0038]

# Outline

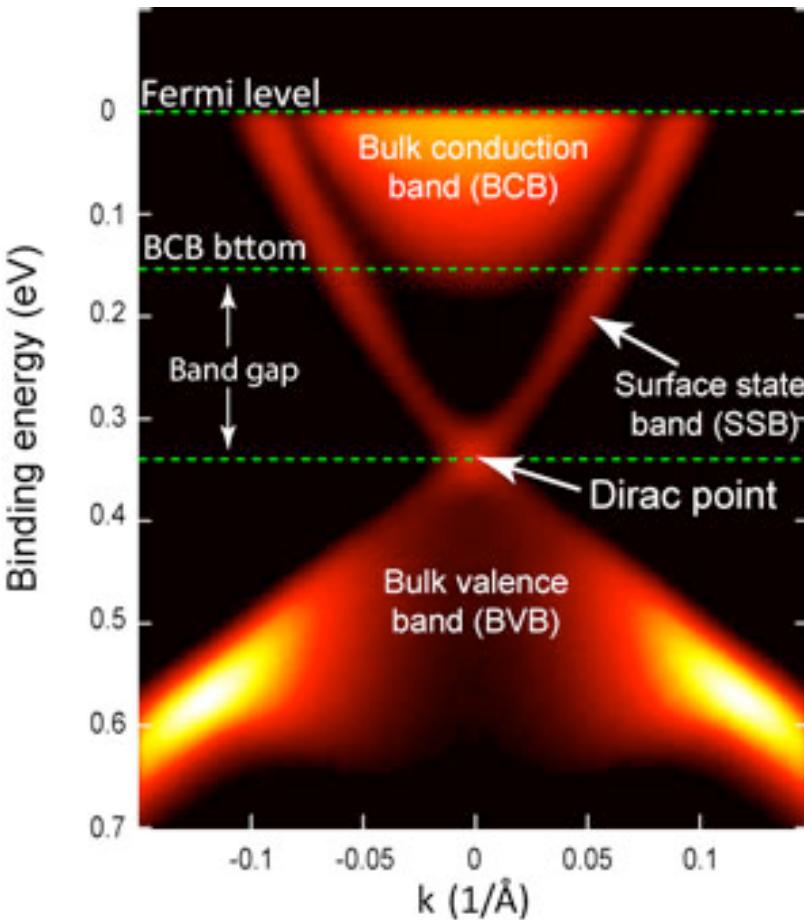
- ✓ Invitation
  - ✓ Review: Hydrodynamics & parity breaking
- II. Parity-violating hydro in 2+1 dimensions
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## II. Parity-odd terms in 2+1 dimensions

### Topological phases in condensed matter

Experiments



Topological insulator  $\text{Bi}_2\text{Se}_3$   
[Chen et al., Science, August 2010]

Theoretical advances

• Berry phases need to be included in Fermi liquid theory if T or inversion broken (e.g. anomalous Hall effect)

[Haldane, PRL 2004]

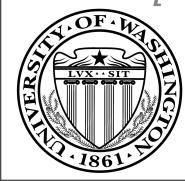
• Hall viscosity

[Avron et al, PRL'94]

[Read, 0805.2507]

[Nicolis & Son, 1103.2137]

[Saremi & Son, 1103.4851]



## II. Parity-odd terms in 2+1 dimensions

### *Basic idea*

Classification: write down all possible terms to first order in gradients

Restrict them by:

- equations of motion ( $0 = \nabla_\mu n u^\mu = n \nabla_\mu u^\mu + u^\mu \nabla_\mu n$ )
- Onsager relations
- two alternatives:
  - positivity of entropy production ( $\nabla_\mu J_s^\mu \geq 0$ )  
or
  - properties of two-point-functions (analyticity, equilibrium relations)



## II. Parity-odd terms in 2+1 dimensions

*What's new in 2+1?*

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \quad \nabla_\mu J^\mu = 0$$

No chiral anomaly, but still a **parity anomaly!**  $J^\mu = J_{\text{even}}^\mu + J_{\text{odd}}^\mu$

Vorticity & magnetic field are pseudoscalars

$$\begin{aligned} \omega^\mu &\longrightarrow \Omega = -\epsilon^{\mu\nu\rho} u_\mu \nabla_\nu u_\rho \\ B^\mu &\longrightarrow B = -\frac{1}{2} \epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho} \end{aligned}$$



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Constitutive equations contain different terms

$$\{u_\mu, T, \mu, F_{\mu\nu}, \nabla_\mu\}$$

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - g_{\alpha\beta} \nabla_\lambda u^\lambda)$$

$$\tilde{U}^\mu = \epsilon^{\mu\nu\rho} u_\nu U_\rho$$

$$\tilde{S}^{\mu\nu} = \frac{1}{2} (\epsilon^{\mu\alpha\rho} u_\alpha S_\rho^\nu + \epsilon^{\nu\alpha\rho} u_\alpha S_\rho^\mu)$$

scalars	pseudoscalars	transverse vectors	tensors
$\nabla_\mu u^\mu$	$\Omega = -\epsilon^{\mu\nu\rho} u_\mu \nabla_\nu u_\rho$ $B = -\frac{1}{2} \epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho}$	$U_1^\mu = u^\alpha \nabla_\alpha u^\mu$ $U_2^\mu = F^{\mu\nu} u_\nu = E^\mu$ $U_3^\mu = \Delta^{\mu\nu} \nabla_\nu \frac{u^\mu}{T} - \frac{E^\mu}{T} = -\frac{V^\mu}{T}$	$\sigma^{\mu\nu}$



## II. Parity-odd terms in 2+1 dimensions

### Classification

Listing all possible (pseudo)tensors, (pseudo)vectors, and (pseudo)scalars

### Decompositions

$$T_{\mu\nu} = \mathcal{E} u_\mu u_\nu + \mathcal{P} \Delta_{\mu\nu} + (q_\mu u_\nu + q_\nu u_\mu) + t_{\mu\nu}$$

$$J_\mu = \mathcal{N} u_\mu + j_\mu$$

$$u_\mu q^\mu = 0, \quad u_\mu t^{\mu\nu} = 0, \quad u_\mu j^\mu = 0$$

Field redefinition  
ambiguity out-of-  
equilibrium

$$\begin{aligned} u_\nu(x) &\rightarrow \hat{u}_\nu(x) \\ T(x) &\rightarrow \hat{T}(x) \\ \mu(x) &\rightarrow \hat{\mu}(x) \end{aligned}$$

Fix by choice of a  
particular  
hydrodynamic frame



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Fix by choice of a particular hydrodynamic frame

Landau frame

$$q_\mu = 0 \quad \mathcal{E} = \epsilon_0 \quad \mathcal{N} = \rho_0$$



$$T^{\mu\nu} = \epsilon_0 u^\mu u^\nu + (P_0 - \zeta \nabla_\alpha u^\alpha - \tilde{\chi}_B B - \tilde{\chi}_\Omega \Omega) \Delta^{\mu\nu} - \eta \sigma^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu}$$

$$J^\mu = \rho_0 u^\mu + \sigma V^\mu + \chi_E E^\mu + \chi_T \Delta^{\mu\nu} \nabla_\nu T + \tilde{\sigma} \tilde{V}^\mu + \tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T$$



## II. Parity-odd terms in 2+1 dimensions

### *Entropy production*

Alternative frame choice

$$T^{\mu\nu} = \epsilon_0 u^\mu u^\nu + P_0 \Delta^{\mu\nu} + \tau^{\mu\nu},$$
$$J^\mu = \rho_0 u^\mu + \Upsilon^\mu, \quad (\text{ideal + corrections})$$



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$$J^\mu = \rho_0 u^\mu + \Upsilon^\mu, \quad (\text{ideal + corrections})$$

Alternative pseudovector basis

transverse pseudovectors
$\tilde{V}_1^\mu = \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T = -T \tilde{U}_1^\mu - R_0 T^2 \tilde{U}_3^\mu$
$\tilde{V}_2^\mu = \tilde{U}_2^\mu$
$\tilde{V}_3^\mu = \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho \frac{\mu}{T} = \tilde{U}_3^\mu + \frac{\tilde{U}_2^\mu}{T}$
$\tilde{V}_4^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = \tilde{U}_2^\mu + u^\mu B$
$\tilde{V}_5^\mu = \epsilon^{\mu\nu\rho} \nabla_\nu u_\rho = -\tilde{U}_1^\mu + u^\mu \Omega$

Most general entropy current

$$J_s^\mu = J_{s \text{ canon}}^\mu + \nu_0 (\nabla \cdot u) u^\mu + \sum_{i=1}^3 \nu_i U_i^\mu + \sum_{i=1}^5 \tilde{\nu}_i \tilde{V}_i$$

$$J_{s \text{ canon}}^\mu = s_0 u^\mu - \frac{\mu}{T} \Upsilon^\mu - \frac{u_\nu}{T} \tau^{\mu\nu}$$



## II. Parity-odd terms in 2+1 dimensions

### *Entropy production*

Structure of divergence

$$\nabla_\alpha J_s^\alpha =$$

+ (products of  
first order data) ,

$$\implies \tilde{\nu}_2 = \nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$$



## II. Parity-odd terms in 2+1 dimensions

### *Entropy production*

Structure of divergence

$$\begin{aligned}\nabla_\alpha J_s^\alpha &= + \left( \nu_2 - \frac{\nu_3}{T} \right) \nabla_\mu E^\mu + \nu_3 \Delta^{\mu\nu} \nabla_\mu \partial_\nu \frac{\mu}{T} \\ &\quad + (\nu_0 + \nu_1) u^\alpha \nabla_\alpha \nabla_\mu u^\mu - \nu_1 u^\alpha u^\mu R_{\alpha\mu} \\ &\quad - \tilde{\nu}_2 u^\alpha \nabla_\alpha B + \left( \begin{array}{c} \text{products of} \\ \text{first order data} \end{array} \right), \\ &\qquad\qquad\qquad \implies \tilde{\nu}_2 = \nu_0 = \nu_1 = \nu_2 = \nu_3 = 0\end{aligned}$$



## II. Parity-odd terms in 2+1 dimensions

### Entropy production

Structure of divergence

$$\begin{aligned}\nabla_\alpha J_s^\alpha &= + \left( \nu_2 - \frac{\nu_3}{T} \right) \nabla_\mu E^\mu + \nu_3 \Delta^{\mu\nu} \nabla_\mu \partial_\nu \frac{\mu}{T} \\ &\quad + (\nu_0 + \nu_1) u^\alpha \nabla_\alpha \nabla_\mu u^\mu - \nu_1 u^\alpha u^\mu R_{\alpha\mu} \\ &\quad - \tilde{\nu}_2 u^\alpha \nabla_\alpha B + (\text{products of first order data}) ,\end{aligned}\Rightarrow \tilde{\nu}_2 = \nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$$

Products of first order data

$$\partial_\alpha J_s^\alpha = + \partial_\alpha J_{s \text{ canon}}^\alpha$$

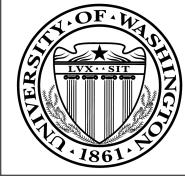
$$- \Omega(\partial \cdot u)$$

$$- B(\partial \cdot u)$$

$$+ U_2 \cdot \tilde{U}_3$$

$$+ U_1 \cdot \tilde{U}_3$$

$$+ U_1 \cdot \tilde{U}_2$$



## II. Parity-odd terms in 2+1 dimensions

### *Entropy production*

Structure of divergence

$$\begin{aligned}
 \nabla_\alpha J_s^\alpha &= + \left( \nu_2 - \frac{\nu_3}{T} \right) \nabla_\mu E^\mu + \nu_3 \Delta^{\mu\nu} \nabla_\mu \partial_\nu \frac{\mu}{T} \\
 &\quad + (\nu_0 + \nu_1) u^\alpha \nabla_\alpha \nabla_\mu u^\mu - \nu_1 u^\alpha u^\mu R_{\alpha\mu} \\
 &\quad - \tilde{\nu}_2 u^\alpha \nabla_\alpha B + \left( \begin{array}{c} \text{products of} \\ \text{first order data} \end{array} \right), \\
 &\qquad\qquad\qquad \implies \tilde{\nu}_2 = \nu_0 = \nu_1 = \nu_2 = \nu_3 = 0
 \end{aligned}$$

Products of first  
order data

$$\begin{aligned}
 \partial_\alpha J_s^\alpha &= + \partial_\alpha J_{s \text{ canon}}^\alpha \\
 &\quad - \Omega(\partial \cdot u) \left[ T \left( \frac{\partial P_0}{\partial \epsilon_0} \right)_{\rho_0} (\partial_T \tilde{\nu}_5 + \tilde{\nu}_1) + \frac{1}{T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\epsilon_0} (\partial_{\bar{\mu}} \tilde{\nu}_5 + \tilde{\nu}_3) \right] \\
 &\quad - B(\partial \cdot u) \left[ T \left( \frac{\partial P_0}{\partial \epsilon_0} \right)_{\rho_0} \partial_T \tilde{\nu}_4 + \frac{1}{T} \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\epsilon_0} \partial_{\bar{\mu}} \tilde{\nu}_4 \right] \\
 &\quad + U_2 \cdot \tilde{U}_3 [R_0 T (\partial_T \tilde{\nu}_3 - \partial_{\bar{\mu}} \tilde{\nu}_1) - \partial_{\bar{\mu}} \tilde{\nu}_4 + R_0 T^2 \partial_T \tilde{\nu}_4] \\
 &\quad + U_1 \cdot \tilde{U}_3 [-R_0 T^2 (\partial_T \tilde{\nu}_5 + \tilde{\nu}_1) + (\partial_{\bar{\mu}} \tilde{\nu}_5 + \tilde{\nu}_3) + T (\partial_{\bar{\mu}} \tilde{\nu}_1 - \partial_T \tilde{\nu}_3)] \\
 &\quad + U_1 \cdot \tilde{U}_2 \left[ \frac{\partial_{\bar{\mu}} \tilde{\nu}_5 + \tilde{\nu}_3}{T} + \partial_{\bar{\mu}} \tilde{\nu}_1 - \partial_T \tilde{\nu}_3 - T \partial_T \tilde{\nu}_4 \right],
 \end{aligned}$$



## II. Parity-odd terms in 2+1 dimensions

### *Entropy production*

Canonical part

$$\begin{aligned} \partial_\alpha J_{s_{\text{canon}}}^\alpha = & - \left( \frac{1}{2} \Delta_{\mu\nu} \tau^{\mu\nu} - \left( \frac{\partial P_0}{\partial \epsilon_0} \right)_{\rho_0} u_\mu u_\nu \tau^{\mu\nu} + \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\epsilon_0} u_\mu \Upsilon^\mu \right) \frac{\partial \cdot u}{T} \\ & - (R_0 u_\mu \tau^{\mu\nu} + \Upsilon^\nu) \Delta_{\nu\alpha} U_3^\alpha \\ & - \frac{\tau^{\mu\nu} \sigma_{\mu\nu}}{2T}. \end{aligned}$$

..... Transform back to Landau frame .....

Thermodynamic response parameters

$$\begin{aligned} \tilde{\chi}_B &= \frac{\partial P_0}{\partial \epsilon_0} \left( T \frac{\partial \mathcal{M}_B}{\partial T} + \mu \frac{\partial \mathcal{M}_B}{\partial \mu} - \mathcal{M}_B \right) + \frac{\partial P_0}{\partial \rho_0} \frac{\partial \mathcal{M}_B}{\partial \mu}, \\ \tilde{\chi}_\Omega &= \frac{\partial P_0}{\partial \epsilon_0} \left( T \frac{\partial \mathcal{M}_\Omega}{\partial T} + \mu \frac{\partial \mathcal{M}_\Omega}{\partial \mu} + f_\Omega(T) - 2\mathcal{M}_\Omega \right) + \frac{\partial P_0}{\partial \rho_0} \left( \frac{\partial \mathcal{M}_\Omega}{\partial \mu} - \mathcal{M}_B \right), \\ \tilde{\chi}_E &= \frac{\partial \mathcal{M}_B}{\partial \mu} - R_0 \left( \frac{\partial \mathcal{M}_\Omega}{\partial \mu} - \mathcal{M}_B \right), \\ T\tilde{\chi}_T &= \left( T \frac{\partial \mathcal{M}_B}{\partial T} + \mu \frac{\partial \mathcal{M}_B}{\partial \mu} - \mathcal{M}_B \right) - R_0 \left( T \frac{\partial \mathcal{M}_\Omega}{\partial T} + \mu \frac{\partial \mathcal{M}_\Omega}{\partial \mu} + f_\Omega(T) - 2\mathcal{M}_\Omega \right), \end{aligned}$$

Matching to two-point functions later gives:  
 (or assumption of “magnetovortical” equilibrium)

$$\mathcal{M}_B = \frac{\partial P}{\partial B}, \quad \mathcal{M}_\Omega = \frac{\partial P}{\partial \Omega}$$



## II. Parity-odd terms in 2+1 dimensions

*Hydrodynamic two-point-functions*

*Simplified example in 2+1 dim:*

$$J^\mu = \rho_0 u^\mu + \sigma E^\mu$$

External sources  $A_t, A_x \propto e^{-i\omega t + ikx}$

$$u^\mu = (1, 0, 0)$$

Allow response  $\rho_0 = \delta\rho$  (fix T and u)



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Allow response  $\rho_0 = \delta\rho$  (fix T and u)

One-point-functions from solving  $\nabla_\mu J^\mu = 0$

$$\langle J^t \rangle = \delta\rho = -\frac{i\sigma k}{\omega + ik^2 \frac{\sigma}{\chi}} (\omega A_x + k A_t)$$

Einstein  
relation for

$$\langle J^x \rangle = \delta\rho = -\frac{i\sigma\omega}{\omega + ik^2 \frac{\sigma}{\chi}} (\omega A_x + k A_t)$$

diffusion:  $D = \frac{\sigma}{\chi}$

$$\langle J^y \rangle = 0$$



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$$\langle J^y \rangle = 0$$

$\Rightarrow$  Two-point-functions  $\langle J^t J^x \rangle = \frac{\delta \langle J^t \rangle}{\delta A_x} = -\frac{i\sigma\omega k}{\omega + iDk^2}$

$\Rightarrow$  Kubo formulae for transport coefficients



## II. Parity-odd terms in 2+1 dimensions

*Hydrodynamic two-point-functions*

Simplified example in 2+1 dim:

$$J^\mu = \rho_0 u^\mu + \sigma E^\mu$$

External sources  $A_t, A_x \propto e^{-i\omega t + ikx}$   
possible: more sources

$$u^\mu = (1, 0, 0)$$

Allow response  $\rho_0 = \delta\rho$  (fix T and u)

generally: T and u respond as well

One-point-functions from solving  $\nabla_\mu J^\mu = 0$

$$\langle J^t \rangle = \delta\rho = -\frac{i\sigma k}{\omega + ik^2 \frac{\sigma}{\chi}} (\omega A_x + k A_t)$$

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## II. Parity-odd terms in 2+1 dimensions

### *Hydrodynamic two-point-functions*

Most general parity-violating case is more complicated

$$\begin{pmatrix} k^2\sigma - i\omega \frac{\partial \rho_0}{\partial \mu} & -k^2 \left( \frac{\mu}{T}\sigma + \chi_T \right) - i\omega \frac{\partial \rho_0}{\partial T} & ik\rho_0 & 0 \\ -i\omega \frac{\partial \epsilon_0}{\partial \mu} & -i\omega \frac{\partial \epsilon_0}{\partial T} & ik(\epsilon_0 + P_0) & 0 \\ ik\rho_0 & iks_0 & k^2(\eta + \zeta) - i\omega(\epsilon_0 + P_0) & k^2(\tilde{\chi}_\Omega + \tilde{\eta}) \\ 0 & 0 & -k^2\tilde{\eta} & k^2\eta - i\omega(\epsilon_0 + P_0) \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \\ \delta u^x \\ \delta u^y \end{pmatrix} = \text{vector containing external sources}$$

$h_{\mu\nu}, A_\mu$

For example, we get a Kubo formula for

$$\lim_{k \rightarrow 0} \frac{1}{ik} \langle \mathcal{C}^0 \mathcal{T}^{02} \rangle_R(0, k) = \tilde{\chi}_\Omega$$



## II. Parity-odd terms in 2+1 dimensions

### *Hydrodynamic two-point-functions*

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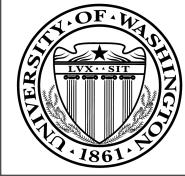
## II. Parity-odd terms in 2+1 dimensions

### *Onsager restrictions*

Restrictions from Onsager relations

$$G_R^{ij}(\omega, \mathbf{k}; b_a) = n_i n_j G_R^{ji}(\omega, -\mathbf{k}; -b_a)$$

where under time-reversal  $\Theta \mathcal{O}_i \Theta^{-1} = n_i \mathcal{O}_i$



## II. Parity-odd terms in 2+1 dimensions

### Onsager restrictions

Restrictions from Onsager relations

$$G_R^{ij}(\omega, \mathbf{k}; b_a) = n_i n_j G_R^{ji}(\omega, -\mathbf{k}; -b_a)$$

where under time-reversal  $\Theta \mathcal{O}_i \Theta^{-1} = n_i \mathcal{O}_i$

From time-reversal covariance plus translation invariance

$$G_R^{ij}(x) \equiv i\theta(t) \operatorname{Tr} (\varrho[\mathcal{O}_i(t, \mathbf{x}), \mathcal{O}_j(0)]) = i\theta(t) n_i n_j \operatorname{Tr} (\varrho'[\mathcal{O}_j(t, -\mathbf{x}), \mathcal{O}_i(0)])$$

Parameters  $b_a$  break time-reversal invariance,  
i.e. time-reversal and  $b_a \rightarrow -b_a$   
together are a symmetry



## II. Parity-odd terms in 2+1 dimensions

### *Equilibrium restrictions*

Restrictions from equilibrium constraints (“susceptibilities”)

*Examples*     $\lim_{\mathbf{k} \rightarrow 0} \langle J^0 J^0 \rangle(\omega = 0, \mathbf{k}) = \left( \frac{\partial \rho_0}{\partial \mu} \right)_T$

Partition function in grand canonical ensemble

$$Z[T, \mu] = \text{Tr} \left[ \exp \left( -\frac{H}{T} + \frac{\mu Q}{T} \right) \right]$$

Constant external sources  $A_0, h_{00}, h_{0i}$   
can be eliminated by shifting thermodynamic variables

$$Z[T, \mu; A_0, h_{00}, h_{0i}] = Z \left[ T \left( 1 + \frac{h_{00}}{2} \right), \mu \left( 1 + \frac{h_{00}}{2} \right) + A_0; 0, 0, 0 \right]$$

Thus we get relations for zero-momentum limits of  
zero-frequency correlators.



## II. Parity-odd terms in 2+1 dimensions

### *Magnetovortical frame*

Thermodynamics depending on vorticity and magnetic field

$$P = P(T, \mu, B, \Omega) \quad dP = s dT + \rho d\mu + \frac{\partial P}{\partial B} B + \frac{\partial P}{\partial \Omega} \Omega, \\ \epsilon + P = sT + \mu\rho.$$

Constitutive relations

$$T^{\mu\nu} = (\epsilon - \mathcal{M}_\Omega \Omega + f_\Omega \Omega) u^\mu u^\nu + (P - \zeta \nabla_\alpha u^\alpha - \tilde{x}_B B - \tilde{x}_\Omega \Omega) \Delta^{\mu\nu} - \eta \sigma^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu}, \\ J^\mu = (\rho - \mathcal{M}_B \Omega) u^\mu + \sigma V^\mu + \tilde{\sigma} \tilde{V}^\mu + \tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T, \\ \text{where } \mathcal{M}_B = \frac{\partial P}{\partial B}, \quad \mathcal{M}_\Omega = \frac{\partial P}{\partial \Omega} \quad \dots + \chi_T \Delta^\mu{}_\nu \nabla_\nu T + \dots$$

Matching

$$\tilde{x}_B = \frac{\partial P}{\partial B}, \quad \tilde{x}_\Omega = \frac{\partial P}{\partial \Omega}, \\ T \tilde{\chi}_T = \frac{\partial \epsilon}{\partial B} + R_0 \left( \frac{\partial P}{\partial \Omega} - \frac{\partial \epsilon}{\partial \Omega} - f_\Omega \right), \quad \tilde{\chi}_E = \frac{\partial \rho}{\partial B} + R_0 \left( \frac{\partial P}{\partial B} - \frac{\partial \rho}{\partial \Omega} \right).$$



## II. Parity-odd terms in 2+1 dimensions

*Hydro without entropy current*

Two-point functions together with “equilibrium correlators“ replace the entropy argument.

*Proven for 2+1 dimensions:*

[Jensen, MK, Kovtun, Meyer, Ritz, Yarom 1112.4498]

*Proven for “equality type” conditions in d dimensions:*

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Inequality type:  $\sigma \geq 0$      $\eta \geq 0$     (from two-point functions)

Generating functional

$$W_m = \int d^d x \mathcal{L}[\text{sources}(x)]$$

⇒ Example: Equality type  $\chi_T = 0$

⇒ Generally: m-point functions,  
simplifies higher order hydro  
(zero frequency)

Example 1: Ideal superfluid

$$W_0 = \int d^d x \sqrt{-g} P(T, \mu, \xi^2)$$



## II. Parity-odd terms in 2+1 dimensions

*Hydro without entropy current*

[Jensen, MK, Kovtun, Meyer, Ritz, Yarom 1203.3556]



Example 1: Ideal superfluid

$$\begin{aligned} W_0 = \int d^d x \sqrt{-g} P(T, \mu, \xi^2) \\ \xi_\mu = -\partial_\mu \phi + A_\mu \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \langle T^{\mu\nu} \rangle &= \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} + f \xi^\mu \xi^\nu, \\ \langle J^\mu \rangle &= \rho u^\mu - f \xi^\mu, \quad u^\mu \xi_\mu = \mu, \end{aligned}$$



## II. Parity-odd terms in 2+1 dimensions

*Hydro without entropy current*

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Example 1: Ideal superfluid

$$W_0 = \int d^d x \sqrt{-g} P(T, \mu, \xi^2) \quad \implies \quad \langle T^{\mu\nu} \rangle = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} + f \xi^\mu \xi^\nu, \\ \xi_\mu = -\partial_\mu \phi + A_\mu \quad \qquad \qquad \qquad \langle J^\mu \rangle = \rho u^\mu - f \xi^\mu, \quad u^\mu \xi_\mu = \mu,$$



Example 2: Parity-violating, 2+1 dimensions

$$W_1 = \int d^3 x \sqrt{-g} \left[ P(\mu, T) - \tilde{\alpha}_1 \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho} - \tilde{\alpha}_2 \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho \right] \\ \implies \chi_E = \chi_T = 0 , \dots$$



## II. Parity-odd terms in 2+1 dimensions

*Hydro without entropy current*

[Jensen, MK, Kovtun, Meyer, Ritz, Yarom 1203.3556]



Example 1: Ideal superfluid

$$W_0 = \int d^d x \sqrt{-g} P(T, \mu, \xi^2) \quad \implies \quad \langle T^{\mu\nu} \rangle = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} + f \xi^\mu \xi^\nu, \\ \xi_\mu = -\partial_\mu \phi + A_\mu \quad \qquad \qquad \qquad \langle J^\mu \rangle = \rho u^\mu - f \xi^\mu, \quad u^\mu \xi_\mu = \mu,$$



Example 2: Parity-violating, 2+1 dimensions

$$W_1 = \int d^3 x \sqrt{-g} \left[ P(\mu, T) - \tilde{\alpha}_1 \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho} - \tilde{\alpha}_2 \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho \right] \\ \implies \chi_E = \chi_T = 0 , \dots$$



Example 3: Parity-preserving, d+1, second order

new relations in d+1,  
reproduces 5 conditions in 3+1  
[Bhattacharyya 1201.4654]



## II. Parity-odd terms in 2+1 dimensions

### Summary of part II

→ Relativistic hydrodynamics was completed at first and second order (Careful with “Causal Viscous Hydro”).

[Baier et al, Minwalla et al 2007]

[Erdmenger, Haack, MK, Yarom 0809.2488]

[Banerjee et al. 0809.2596]

→ Chiral transport effects measured in heavy-ion-collisions?

[Kharzeev, Son]

→ New methods for hydrodynamic correlation functions

→ New method restricting transport coefficients

Note also: Relation of chiral vortex & magnetic coefficients with anomalies can also be derived from such an equilibrium functional Ward identities using zero-frequency two- and three-point-functions!

[Jensen 1203.3599]



# Outline

- ✓ Invitation
- ✓ Review: Hydrodynamics & parity breaking
- ✓ Parity-violating hydro in 2+1 dimensions
  - Classify transport coefficients
  - Restrictions from entropy production
  - Restrictions from two-point-functions

III. One gravity dual

IV. Conclusions



### **III. Gravity dual**

*-Basic idea*

Equilibrium solution confirms thermodynamics  
with vorticity and magnetic field.

Fluid/gravity approach confirms our constitutive  
relations.



# III. Gravity dual

Gauge/Gravity Dictionary

Fluid/Gravity Correspondence (on boundary)

[Baier et al. 2007]

[Bhattacharyya et al. 0712.2456]

[cf. talk by Keeler]

$$\text{Einstein equations} = \begin{array}{l} \text{hydrodyn. conservation} \\ = \end{array} \begin{array}{l} \text{dynamical} \\ + \text{EOMs for} \\ \text{equations} \end{array} \begin{array}{l} \text{gravity fields} \end{array}$$



Gauge Theory

Temperature



Gravity Theory

Hawking

$T \sim$  horizon radius

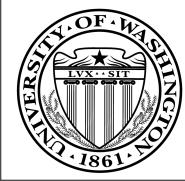
Thermal QFT  
with charge



Charged  
black hole

$$g_{\mu\nu}(r; r_{\text{Horizon}})$$

$$g_{\mu\nu}(r, t, \vec{x}; T, \mu, u^\mu)$$



# III. Gravity dual

Gauge/Gravity Dictionary

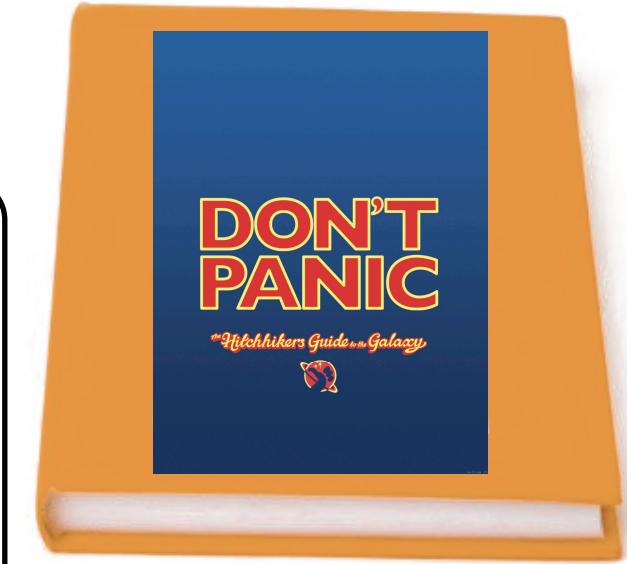
Fluid/Gravity Correspondence (on boundary)

[Baier et al. 2007]

[Bhattacharyya et al. 0712.2456]

[cf. talk by Keeler]

Einstein equations = hydrodyn. conservation + EOMs for equations      dynamical gravity fields



Gauge Theory

Temperature



Gravity Theory

Hawking

T ~ horizon radius

Thermal QFT  
with charge



Charged  
black hole

$g_{\mu\nu}(r; r_{\text{Horizon}})$

$g_{\mu\nu}(r, t, \vec{x}; T, \mu, u^\mu)$

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu - 2u_\mu dx^\mu dr$$

Promote parameters and expand  
in gradients:  $T(t, \vec{x})$ ,  $\mu(t, \vec{x})$ ,  $u^\mu(t, \vec{x})$



# III. Gravity dual

Gravity action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{4}F^2 - \frac{1}{2}(\partial\varphi)^2 - V[\varphi] \right] - \frac{1}{64\pi^2} \int d^4x \theta[\varphi] \epsilon^{abcd} F_{ab} F_{cd} + S_{\text{bdy}}$$

Equations of motion

$$\begin{aligned} R_{ab} - \frac{R}{2}g_{ab} &= \frac{3}{L^2}g_{ab} + \tau_{ab}, \\ \partial_a \left( \sqrt{-g}F^{ab} \right) &= \frac{\kappa^2}{8\pi^2} \epsilon^{bcde} \partial_c \theta F_{de}, \\ \partial_a \left( \sqrt{-g}\partial^a \varphi \right) &= \sqrt{-g}V'[\varphi] + \frac{\theta'[\varphi]\kappa^2}{32\pi^2} \epsilon^{abcd} F_{ab} F_{cd} \end{aligned}$$

Ansatz

$$\begin{aligned} \delta g_{ab} dx^a dx^b &= 2\Omega r^2 f(r) x^1 dt dx^2 - \Omega x^1 dr dx^2 + \mathcal{O}(J_\varphi^2), \quad f(r) = 1 - \frac{r_H^3}{r^3} + \mathcal{O}(\mu^2, J_\varphi^2) \\ \delta A_a dx^a &= \left( B + \frac{\mu r_H \Omega}{r} \right) x^1 dx^2 + \mathcal{O}(J_\varphi^2), \end{aligned}$$



### III. Gravity dual

Equilibrium solutions **confirm our magnetovortical thermodynamics**

$$\begin{aligned}\frac{\partial \epsilon}{\partial B} &= T \frac{\partial \mathcal{M}_B}{\partial T} + \mu \frac{\partial \mathcal{M}_B}{\partial \mu} - \mathcal{M}_B, \\ \frac{\partial \epsilon}{\partial \Omega} &= T \frac{\partial \mathcal{M}_\Omega}{\partial T} + \mu \frac{\partial \mathcal{M}_\Omega}{\partial \mu} - \mathcal{M}_\Omega - f_\Omega(T),\end{aligned}$$

$$\begin{array}{ll} \frac{\partial \epsilon}{\partial \Omega} = (\Delta - 2)\mu^2 r_H \Phi_3 - f_\Omega(T) + \mathcal{O}(\mu^3, J_\varphi^2) & \frac{\partial \epsilon}{\partial B} = (\Delta - 3)\mu \Phi_2 + \mathcal{O}(\mu^2, J_\varphi^2) \\ \frac{\partial \rho}{\partial \Omega} = 2\mu r_H \Phi_3 + \frac{\mu \theta_0}{8\pi^2} + \mathcal{O}(\mu^2, J_\varphi^2) & \frac{\partial \rho}{\partial B} = \Phi_2 + \frac{\theta_0}{8\pi^2} + \mathcal{O}(\mu^2, J_\varphi^2) \\ \frac{\partial P}{\partial \Omega} = \mathcal{M}_\Omega = \mu^2 r_H \Phi_3 + \frac{\mu^2 \theta_0}{16\pi^2} + c_\Omega(T) + \mathcal{O}(\mu^3, J_\varphi^2), & \frac{\partial P}{\partial B} = \mathcal{M}_B = \mu \Phi_2 + \frac{\mu \theta_0}{8\pi^2} + c_B(T) + \mathcal{O}(\mu^2, J_\varphi^2) \end{array}$$

Fluid/gravity result **confirms form of constitutive relations** with

$$\begin{array}{lll} \tilde{\eta} = 0 & \sigma = \frac{1}{2\kappa^2} + \mathcal{O}(\mu^2, J_\varphi^2), & \tilde{\sigma} = \frac{\theta_1}{8\pi^2} \varphi(r_H) - \frac{\partial \rho}{\partial B} + \mathcal{O}(\mu^2, J_\varphi^2), \\ \tilde{\chi}_E = \frac{\partial \rho}{\partial B} + \mathcal{O}(\mu^2, J_\varphi^2), & & T \tilde{\chi}_T = \frac{\partial \epsilon}{\partial B} + \mathcal{O}(\mu^3, J_\varphi^2), \end{array}$$



## IV. Conclusions

- ✓ completed first order hydro in 2+1
- ✓ new methods for restricting transport coefficients
- ✓ “new” transport effects
- application to surface states in topological insulators, or to graphene, or ...?
- bound currents, non-relativistic?
- equilibrium with vorticity, “ferrovorticism”

