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Introduction to String Theory

Exercises III

Ex. 3.1 Polchinski exercise 4.2

Hint: Recall that for a highest weight state $L_n^m|\chi\rangle = 0$, and for a physical state $(L_n^m + A^m\delta_{n,0})|\phi\rangle = 0$.

Ex. 3.2 Compactifications, T-duality & S-duality

Recall that T-duality and S-duality relate various string theories to each other. They are thus a vital ingredient for our understanding of the *terra incognita* of M-theory.

T-duality in bosonic string theory compactified on a circle with radius R in the 25^{th} dimension is a symmetry of the bosonic string solution under a particular transformation. This is the transformation of the compactification radius $R \to \tilde{R} = {l_s}^2/R$ and simultaneous interchange of the winding number W with the Kaluza-Klein excitation number K. Therefore, bosonic string theory compactified on a circle with radius R with W windings around that circle and with momentum $p^{25} = K/R$ is equivalent to a bosonic string theory compactified on a circle with radius ${l_s}^2/R$ with winding number K and momentum $p^{25} = W/R$. To see this in more detail, consider the closed bosonic string action in 25-dimensional bosonic string theory with target space coordinates X^μ [1]

$$S_{\text{bosonic}} = -T \int d\sigma d\tau \sqrt{-\det g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu}\,, \tag{1}$$

with the metric g, the string tension T and a $1+1$-dimensional parametrization ($\sigma^0 = \tau, \sigma^1 = \sigma$) of the world sheet where α, $\beta = 0, 1$. Here the parameters are the worldsheet time $\tau = 0, \dots, 2\pi$ and spatial coordinate $\sigma = 0, \dots, \pi$. Note, that we could generalize this action (1) to the case of a simple p-dimensional object, a *Dp-brane*. The most general solution is given by the sum of one solution in which the modes travel in one direction on the closed string (*left-movers*) and the second solution where the modes travel in the opposite direction (*right-movers*)

$$X^\mu = X^\mu_L + X^\mu_R\,, \tag{2}$$

which for closed strings are given by

$$X^\mu_L = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2 p^\mu(\tau - \sigma) + \tfrac{i}{2}l_s \sum_{n\neq 0} \tfrac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \tag{3}$$

$$X^\mu_R = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2 p^\mu(\tau + \sigma) + \tfrac{i}{2}l_s \sum_{n\neq 0} \tfrac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}\,. \tag{4}$$

These solutions each consist of three parts: the center of mass position term, the total string momentum or *zero mode* term and the string excitations given by the sum.

a) What do you get if you compactify the 25^{th} dimension on a circle with radius R?

Answer:

$$
\begin{aligned}
X_L^{25} &= \tfrac{1}{2}(x^{25} + \tilde{x}^{25}) + (\alpha' p^{25} + WR)(\tau + \sigma) + \dots && (5) \\
X_R^{25} &= \tfrac{1}{2}(x^{25} - \tilde{x}^{25}) + (\alpha' p^{25} - WR)(\tau - \sigma) + \dots , && (6)
\end{aligned}
$$

We leave out the sum over excitation modessince it is invariant under compactification. Only the *zero mode* is affected by the compactification since the momentum becomes $p^{25} = K/R$ with K labeling the levels of the Kaluza-Klein tower of excitations becoming massive upon compactification. An extra winding term is added as well.

b) So what is the sum of both solutions in 25-direction?

Answer:

$$
X^{25} = x^{25} + 2\alpha' \tfrac{K}{R}\tau + 2WR\sigma + \dots . \quad (7)
$$

We now see explicitly that the transformation $W \leftrightarrow K$, $R \to \alpha/R$ applied to equations (5) and (6) is a symmetry of this theory because the *zero mode* changes as $(\alpha' K/R \pm WR) \to (\alpha' WR/\alpha' \pm K\alpha'/R) = (WR \pm \alpha' K/R)$. So we get the transformed solution

$$
\tilde{X}^{25} = \tilde{x}^{25} + 2WR\tau + 2\alpha' \tfrac{K}{R}\sigma + \dots . \quad (8)
$$

Comparing the solutions (8) and (7) we note that the transformed solution is equal to the original one except for the fact that σ and τ are interchanged. However, the bosonic string action is reparametrization invariant * under $(\tau, \sigma) \to (\tilde{\tau}, \tilde{\sigma})$. Therefore we see that physical quantities like correlation functions are invariant under the T-duality tranformation.

From this duality we learn how we may start from one string theory and by different ways of compactification we arrive at two distinct but equivalent formulations of the same physics. Another important feature is that certain quantities change their roles as we go from one compactification to the other (winding modes turn into Kaluza-Klein modes as $K \leftrightarrow W$). Finally we realize that T-duality relates a theory compactified on a large circle R to a theory compactified on a small circle α'/R.

By virtue of T-duality another important ingredient for the gauge/gravity correspondence was introduced into string theory: Dp-branes. Introducing open strings into the bosonic theory of closed strings, we need to specify boundary conditions at the string end points. A natural criterion for these boundary conditions is to preserve Poincaré invariance. So we would choose Neumann boundary conditions $\partial_\sigma X^\mu = 0$ at the end points $\sigma = 0, \pi$. Evaluating this condition for the general solution given in (7), we see that the Neumann condition turns into a Dirichlet boundary condition $\partial_\tau X^\mu = 0$. This condition explicitly breaks Poincaré invariance by fixing p of the spatial coordinates of open string ends to τ-independent hypersurfaces. These surfaces are called Dirichlet- or Dp-branes and have to be considered as dynamical objects in addition to the fundamental strings. We will see below that AdS/CFT is a duality arising from two distinct ways of describing these Dp-branes in open string theory.

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*S-duality exchanges the fundamental strings (i.e. the NS-NS or the Ramond-Ramond two-forms) with the D1-branes. So, roughly speaking the string behaves like a D1-brane. Generalizing the case $p = 1$ to arbitrary p we would find that the Dp-brane action is reparametrization invariant under a change of the $p+1$ world-volume coordinates given by $\sigma^\alpha \to \sigma^\alpha(\tilde{\sigma})$.

Analogous to T-duality, S-duality relates a string theory with coupling constant g_s to a string theory with coupling $1/g_s$.

Ex 3.3 is a HOMEWORK TASK: Spinors & supersymmetry

Read appendix B.1 and B.2 (in particular the remarks on supergravity therein).

References

[1] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: A modern introduction," Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p