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Introduction to Gauge/Gravity & Heavy-Ion-Applications

Exercises II

Ex. 2.1 Relativistic Hydrodynamics in d dimensions

The energy-momentum tensor for a dissipative fluid is given by

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu} \,.$$

The dissipative part $\Pi^{\mu\nu}$ may be written as

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu} \,,$$

where η is the shear viscosity and ζ the bulk viscosity. We have the velocity u^μ, energy density ϵ, pressure P. The traceless symmetric tensor $\sigma^{\mu\nu}$ and the trace part θ are defined as

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\nabla_{(\alpha}u_{\beta)} - \frac{1}{d-1}\theta P^{\mu\nu} \,,$$
$$\theta = \nabla_\mu u^\mu = P^{\mu\nu}\nabla_\mu u_\nu \,,$$

where $P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$.

a) Show that $P^{\mu\nu}$ is the projection operator onto directions perpendicular to u^μ, i. e. prove $P^{\mu\nu}u_\nu = 0$, $P^{\mu\alpha}P_{\alpha\nu} = P^\mu_\nu = P^{\mu\alpha}g_{\alpha\nu}$ and $P^\mu_\mu = d-1$.

b) Show that the entropy current is not conserved and satisfies

$$\nabla_\mu J^\mu_s = \frac{2\eta}{T}\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \frac{\zeta}{T}\theta^2 \,.$$

What is the implication of this equation on the sign of η and ζ?

Hint: You may use the identity $0 = T\nabla_\mu(su^\mu) + \Pi^{\mu\nu}\nabla_\mu u_\nu$. Additionally, show that the energy momentum-tensor satisfies the Landau frame condition $u_\nu T^{\mu\nu} = -\epsilon u^\mu$. This implies $u_\nu \sigma^{\mu\nu} = 0$ which is needed in the calculation.

Ex. 2.2 Correlators & Thermal Spectral Functions

Remark: There is probably not enough time to derive every step in this exercise. It may be useful to work along (arXiv:hep-th/0205052, and possibly also arXiv:0808.1114).

Apply the recipe introduced in the lecture in order to compute some correlation functions in $\mathcal{N} = 4$ SYM theory with R-charge current J^μ (dual to the five-dimensional vector field A_μ on the gravity side).

The part of the action quadratic in the gauge field A is given by

$$S^{(2)} = -\frac{N^2}{16\pi^2}\int du\, d^4x \sqrt{-g(u)} F_{\mu\nu}F^{\mu\nu} \,. \tag{1}$$

In order to place our field theory at finite temperature, we will work in the dual AdS black hole background

$$ds^2 = \frac{(\pi T R)^2}{u}[-f(u)dt^2 + d\mathbf{x}^2] + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5{}^2 \,, \qquad (2)$$

with the radial AdS-coordinate $u \in [0,1]$, the horizon at $u = 1$, spatial infinity at $u = 0$ and the function $f(u) = 1 - u^2$. This metric is obtained from the standard AdS black hole metric with radial coordinate r by the transformation $u = (r_0/r)^2$. The temperature $T = r_0/(\pi R^2)$ is a function of the AdS-radius R and the black hole horizon r_0.

a) Derive the equations of motion for the gauge field components in Fourier-space using

$$A_i(u, \vec{x}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} A_i(u, \vec{k}) \,. \qquad (3)$$

b) Find the *indicial exponents* which regularize the regular singular coefficients at the horizon.

c) Expand the function into powers of ω and q^2, plug this into the equation of motion and expand the resulting expression into powers of ω and q^2. Solve this order by order.

d) Compute the correlation functions $\langle J_t J_t \rangle$ and $\langle J_x J_x \rangle$.

Ex. 2.3 Viscosity Bound

a) Find a numerical solution to the equation

$$0 = \phi'' - \frac{1+u^2}{uf}\phi' + \frac{\omega^2 - q^2 f}{uf^2}\phi \,, \qquad (4)$$

with $f = (1 - u^2)$. Use the ingoing boundary condition at the horizon $u = 1$, and choose an arbitrary normalization ($\phi(u = 1) = 1$ may be convenient).

This equation arises as the equation of motion for the off-diagonal (shear) metric perturbation h_{xy} in $\mathcal{N} = 4$ SYM. This perturbation is holographically dual to (in other words *its boundary value sources*) the energy momentum tensor component T_{xy} in the dual gauge theory. Let us set $\vec{q} = 0$.

b) Compute the two-point correlation function $G^R_{xy,xy}(\omega, \vec{0}) = \langle T_{xy} T_{xy} \rangle(\omega, \vec{0})$, and plot the thermal spectral function against frequency ω.

c) Use the following Kubo formula in order to numerically compute the shear viscosity

$$\eta = -\lim_{\omega \to 0} \frac{1}{2\omega} \mathrm{Im} G^R_{xy,xy}(\omega, \vec{q} = 0) \,. \qquad (5)$$

d) In which sense is this approach more powerful than the hydrodynamic one?