Matthias Kaminski

Introduction to Gauge/Gravity & Heavy-Ion-Applications

Exercises I

Ex. 1.1 AdS coordinates

Let us imagine a $(d+2)$-dimensional flat space. Now embed a curved space into this flat space, for example a Lorentzian AdS_{d+1}. It can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 , \tag{1}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} dX^a dX^b$ with $\eta = \text{diag}(-1, 1, 1, \dots, 1, -1)$. In the following we parametrize the locus (1) in different ways.

a) As an example draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$!

b) Compute the curvature $\mathcal{R}$ (Riemann scalar) of AdS_2. This space has the metric

$$ds^2 = \frac{L^2}{r^2}\left(dr^2 - dt^2 + dx^2\right) . \tag{2}$$

Use Mathematica with the package diffgeo.m, or an equivalent program and/or package for differential geometry.

Ex. 1.2 Brane embeddings

The near-horizon limit of a stack of N_c black D3-branes generates a geometry with the metric

$$ds^2 = \frac{1}{2}\left(\frac{\varrho}{L}\right)^2 \left(-\frac{f^2}{\tilde{f}} dt^2 + \tilde{f} d\vec{x}^2\right) + \left(\frac{L}{\varrho}\right)^2 \left(d\varrho^2 + \varrho^2 d\Omega_5^2\right) , \tag{3}$$

with the functions

$$f = (1 - \frac{\varrho_H^4}{\varrho^4}) , \qquad \tilde{f} = (1 + \frac{\varrho_H^4}{\varrho^4}) \tag{4}$$

The function $f = (1 - \frac{\varrho_H^4}{\varrho^4})$ shows the presence of a horizon since it makes the metric component g_{tt} vanish at a finite radius ϱ_H, i.e. $g_{tt}(\varrho_H) = 0$. This horizon is located at $\varrho_H = T\pi L^2$, where T is the Hawking temperature which coincides with the temperature of the dual thermal field theory. L is the radius of the AdS space.

This metric (3) reduces to the $AdS_5 \times S^5$ black hole metric known from the lecture with the coordinate transformation $\varrho^2 = u^2 + \sqrt{u^4 - u_H^4}$.

We prepare the metric by writing out some of the S^5-coordinates namely

$$d\varrho^2 + \varrho^2 d\Omega_5^2 = d\varrho^2 + \varrho^2 \left(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\Omega_3^2\right) . \tag{5}$$

Now we embed a D7-brane into this geometry. It shares the four Minkowski-coordinates (t, x, y, x) with the D3-branes, but also wraps four additional coordinate directions as indicated in the table:

	0	1	2	3	4	5	6	7	8	9
D3	X	X	X	X						
D7	X	X	X	X	X	X	X	X		

a) Compute the metric G which is induced on the D7 brane.

The world-volume theory on the D7-brane is described by the Dirac-Born-Infeld (DBI) action

$$S_{D7} = -N_f T_{D7} \int d^8\xi \sqrt{-G}\,, \tag{6}$$

$$= -N_f T_{D7} \varrho_H^3 \int d^8\xi \frac{\rho^3}{4} f\tilde{f}(1-\chi^2)\sqrt{1-\chi^2+\rho^2\chi'^2}\,, \tag{7}$$

with the brane tension T_{D7}, the dimensionless radial coordinate $\rho = \varrho/\varrho_H$, the *embedding function* $\chi(\rho) = \cos\theta(\rho)$ and the induced metric G.

Varying this DBI-action with respect to the field χ, we get its Euler-Lagrange equation

$$0 = \partial_\rho \left[\frac{\rho^5 f\tilde{f}(1-\chi^2)\chi'}{\sqrt{1-\chi^2+\rho^2\chi'^2}} \right] + \frac{\rho^3 f\tilde{f}\chi}{\sqrt{1-\chi^2+\rho^2\chi'^2}} \left[3(1-\chi^2) + 2\rho^2\chi'^2 \right] \tag{8}$$

b) Find the three leading terms in χ as a function of ρ near the horizon $\rho = 1$.

c) Find a numerical solution for equation (8), and plot that solution against the radial coordinate ρ.

Hint: You may use a shooting method.

*d) Extract the quark mass from the near-boundary behavior of the numerical solution and plot it versus the horizon-parameter χ_0.